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1. The Fundamental Identity.

Let $F(\mathbf{x})$ denote a homogeneous polynomial of degree K in N variables $x_1, x_2, \cdots x_N$. Let the number of terms in this polynomial be n, that is

$$n = (K+1)(K+2)(K+3)\cdots(K+N-1)/1.2.3.\cdots(N-1). \quad (1)$$

Let $F(\mathbf{a}_i)$ be the value of the polynomial when a set of values a_{i1}, $a_{i2}, \cdots a_{iN}$ is assigned to the variables $x_1, x_2, \cdots x_N$. We say that $F(\mathbf{a}_i)$ is the value of the polynomial at the point $\mathbf{a}_i$.

As a temporary notation, let the coefficients of the various terms of the polynomial be $A_1, A_2, \cdots A_n$, with the terms written in some determined order. We may suppose this order to be descending order of x_1, x_2, etc., meaning that the term containing $x_1{}^K$ is the leading term, followed by all terms containing $x_1{}^{K-1}$, then all terms containing $x_1{}^{K-2}$ and so on; and that in each of these groups we follow descending order of x_2 by a similar rule, and so on. Thus

$$F(\mathbf{x}) = A_1x_1{}^K + A_2x_1{}^{K-1}x_2 + A_3x_1{}^{K-1}x_3 + \cdots + A_nx_N{}^K. \quad (2)$$

Now let there be a set of n points $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_i \cdots \mathbf{a}_n$. If we expand the value of the polynomial at each of these points in the form (2) we shall have n equations of the form

$$F(\mathbf{a}_i) = A_1a_{i1}{}^K + A_2a_{i1}{}^{K-1}a_{i2} + A_3a_{i1}{}^{K-1}a_{i3} + \cdots + A_na_{iN}{}^K, \quad (3)$$

which, together with (2), constitutes a set of $n + 1$ linear homogeneous equations in the $n + 1$ quantities $A_1, A_2, \cdots A_n$ and unity. It is accordingly evident that the determinant

$$\begin{vmatrix} F(\mathbf{x}), & x_1{}^K, & x_1{}^{K-1}x_2, & \cdots, & x_N{}^K \\ F(\mathbf{a}_1), & a_{11}{}^K, & a_{11}{}^{K-1}a_{12}, & \cdots, & a_{1N}{}^K \\ F(\mathbf{a}_2), & a_{21}{}^K, & a_{21}{}^{K-1}a_{22}, & \cdots, & a_{2N}{}^K \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F(\mathbf{a}_n), & a_{n1}{}^K, & a_{n1}{}^{K-1}a_{n2}, & \cdots, & a_{nN}{}^K \end{vmatrix} \tag{4}$$

must vanish. Developing by the elements of the first column we may write

$$F(\mathbf{x})C_0 + F(\mathbf{a}_1)C_1 + F(\mathbf{a}_2)C_2 + \cdots + F(\mathbf{a}_n)C_n = 0 \tag{5}$$

where C_0, C_1, etc. are the cofactors of the elements into which they are multiplied.

If $\mathbf{x}$ be variable, with $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$ constant, the identity (5) will be the expansion of the polynomial $F(\mathbf{x})$ in terms of its values at n fixed points, provided C_0 does not vanish. The coefficients $C_1, C_2, \cdots C_n$ are polynomials of degree K in $\mathbf{x}$. They are also polynomials of degree K in each of the fixed points with exception of the point having the same subscript: C_i is independent of $\mathbf{a}_i$. On the other hand C_0 is a constant in the sense of being independent of $\mathbf{x}$. The C's are all independent of the special choice of the polynomial $F(\mathbf{x})$, that is of $A_1, A_2, \cdots A_n$.

Let us next take $\mathbf{e}_1, \mathbf{e}_2, \cdots \mathbf{e}_N$ a system of linearly independent vectors. Any point $\mathbf{x}$ may be regarded as a point vector,

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_N\mathbf{e}_N \tag{6}$$

and in a similar manner

$$\mathbf{a}_i = a_{i1}\mathbf{e}_1 + a_{i2}\mathbf{e}_2 + \cdots + a_{iN}\mathbf{e}_N. \tag{7}$$

Let us furthermore take a system of N polynomials of degree K, namely $F_1(\mathbf{x}), F_2(\mathbf{x}), \cdots F_N(\mathbf{x})$, and write

$$\mathbf{F}(\mathbf{x}) = \mathbf{e}_1F_1(\mathbf{x}) + \mathbf{e}_2F_2(\mathbf{x}) + \cdots + \mathbf{e}_NF_N(\mathbf{x}). \tag{8}$$

By definition, therefore, $\mathbf{F}(\mathbf{x})$ is a vector function of the point vector $\mathbf{x}$, homogeneous of degree K in the variables $x_1, x_2, \cdots x_N$. Vector functions are distinguished by bold-faced type.

Each of the scalar functions which are thus assigned as components

of the vector $\mathbf{F}(\mathbf{x})$ will satisfy an identity of the form (5). We have then a set of N identities

$$\begin{array}{l} F_1(\mathbf{x})C_0 + F_1(\mathbf{a}_1)C_1 + F_1(\mathbf{a}_2)C_2 + \cdots + F_2(\mathbf{a}_n)C_n = 0 \\ F_2(\mathbf{x})C_0 + F_2(\mathbf{a}_1)C_1 + F_2(\mathbf{a}_2)C_2 + \cdots + F_2(\mathbf{a}_n)C_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_N(\mathbf{x})C_0 + F_N(\mathbf{a}_1)C_1 + F_N(\mathbf{a}_2)C_2 + \cdots + F_N(\mathbf{a}_n)C_n = 0 \end{array}$$

wherein we assume that the fixed points $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$, while wholly arbitrary, are constant for all the identities of the set. Hence the coefficients $C_0, C_1, C_2, \cdots C_n$ are unaltered as we pass from one identity to another. If, then, we multiply these identities in turn by $\mathbf{e}_1, \mathbf{e}_2, \cdots \mathbf{e}_N$, and add the results, we shall have the vector identity

$$\mathbf{F}(\mathbf{x})C_0 + \mathbf{F}(\mathbf{a}_1)C_1 + \mathbf{F}(\mathbf{a}_2)C_2 + \cdots + \mathbf{F}(\mathbf{a}_n)C_n = 0. \qquad (9)$$

It is obvious that the left member of this identity may, if we wish, be written in the form of the determinant (4), the elements of the first column being now vectors.

Just as with the scalar identity (5), we may regard the vector identity (9) in two ways,— either as the statement of the linear relation connecting the values of the vector polynomial $\mathbf{F}(\mathbf{x})$ at $n + 1$ arbitrary points; or as the expansion of this polynomial in terms of its values at n points arbitrary except that C_0 does not vanish. The vector function $\mathbf{F}(\mathbf{x})$ depends on N scalar polynomials, each containing n arbitrary numerical coefficients. Whenever these coefficients need to be separately considered they may be designated according to the matrix

$$\begin{array}{lllll} A_{11}, & A_{21}, & A_{31}, & \cdots, & A_{n1} \\ A_{12}, & A_{22}, & A_{32}, & \cdots, & A_{n2} \\ A_{13}, & A_{23}, & A_{33}, & \cdots, & A_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1N}, & A_{2N}, & A_{3N}, & \cdots, & A_{nN} \end{array}$$

Thus the set of numbers A_{ij}, like the set a_{ij}, contains Nn elements.

We may now state the following simple but fundamental theorem:

Theorem I. *The values taken on by an arbitrary vector polynomial at* $\mathrm{n} + 1$ *arbitrary points are linearly related. The multipliers* $\mathrm{C}_0, \mathrm{C}_1, \mathrm{C}_2, \cdots \mathrm{C}_n$ *are functions of the arbitrary points, but are independent of the coefficients* A_{ij} *of the terms of the polynomial.*

2. Geometric Meaning of the Coefficients.

To take a specific example, let $K = 1$ and $N = 3$. The vector determinant of the form (4) becomes

$$\begin{vmatrix} \mathbf{F}(\mathbf{x}) , & x_1 , & x_2 , & x_3 \\ \mathbf{F}(\mathbf{a}_1) , & a_{11}, & a_{12}, & a_{13} \\ \mathbf{F}(\mathbf{a}_2) , & a_{21}, & a_{22}, & a_{23} \\ \mathbf{F}(\mathbf{a}_3) , & a_{31}, & a_{32}, & a_{33} \end{vmatrix}$$

The coefficients C_0, C_1, C_2, and C_3 become determinants of the components of three vectors. If we denote these determinants by (123), $(x23)$, $(x13)$, and $(x12)$, the identity (9) becomes

$$\mathbf{F}(\mathbf{x})\,(123) - \mathbf{F}(\mathbf{a}_1)\,(x23) + \mathbf{F}(\mathbf{a}_2)\,(x13) - \mathbf{F}(\mathbf{a}_3)\,(x12) = 0, \quad (10)$$

a relation very familiar to students of vector analysis, $\mathbf{F}(\mathbf{x})$ being now a linear vector function. When $K = 1$ a similar relation holds for any value of N, whence clearly the vanishing of any C means coplanarity of the vectors which enter into it.

Next take $K = 2$ and $N = 3$. The vector determinant of the form (4) becomes

$$\begin{vmatrix} \mathbf{F}(\mathbf{x}) , & x_1^2 , & x_1x_2 , & x_1x_3 , & x_2^2 , & x_2x_3 , & x_3^2 \\ \mathbf{F}(\mathbf{a}_1) , & a_{11}^2 , & a_{11}a_{12} , & a_{11}a_{13} , & a_{12}^2 , & a_{12}a_{13} , & a_{13}^2 \\ \mathbf{F}(\mathbf{a}_2) , & a_{21}^2 , & a_{21}a_{22} , & a_{21}a_{23} , & a_{22}^2 , & a_{22}a_{23} , & a_{23}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{F}(\mathbf{a}_6) , & a_{61}^2 , & a_{61}a_{62} , & a_{61}a_{63} , & a_{62}^2 , & a_{62}a_{63} , & a_{63}^2 \end{vmatrix} \quad (11)$$

In this case the vanishing of one of the coefficients C means that the six vectors which enter into it lie on a quadric cone, (which may, however, be a degenerate cone). For the equation $C_i = 0$ is homogeneous of the second degree in six vectors, and holds true when any two of them coincide, because two rows of the above determinant become equal. Therefore if, as usual, we regard $\mathbf{x}$ as a variable vector, the equation $C_i = 0$, for values of the subscript other than zero, is the equation of the cone determined by the five fixed vectors which enter into it. Similar reasoning applies for any values of N and K.

3. The Coefficients as Eliminants of K-adics.

The coefficients C may be interpreted in another way, so as to connect them with the theory of matrices and allied operators. We take,

as before, a set of n vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$. Consider for a moment the case $K = 2$. A symmetrical dyadic may be written as a sum of terms of the form $\mathbf{aa}$ where $\mathbf{a}$ is a vector. Let there be a set of constants $c_1, c_2, \cdots c_n$ not all zero. We define the equation

$$c_1\mathbf{a}_1\mathbf{a}_1 + c_2\mathbf{a}_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n\mathbf{a}_n = 0 \tag{12}$$

to be equivalent to the set of n scalar equations obtained by selecting pairs of corresponding components in all possible ways:—

$$c_1a_{1i}a_{ij} + c_2a_{2i}a_{2j} + \cdots + c_na_{ni}a_{nj} = 0 \tag{13}$$

where the subscripts i and j may have any values from 1 to N. These equations are linear and homogeneous in the c's; hence the determinant whose elements are $a_{hi}a_{hj}$ must vanish if (12) holds true. A glance at (11) shows that this determinant is the same as C_0, the minor of $\mathbf{F}(\mathbf{x})$,— by interchanging rows and columns. Therefore the necessary and sufficient condition for the existence of a dyadic equation of the form (12) is that the n vectors which enter in that equation should lie on a quadric cone.

A symmetrical K-ad should properly be written

$$\mathbf{aa} \cdots \text{to } K \text{ factors} \tag{14}$$

but, when no ambiguity is brought about, we may write it as $\mathbf{a}^K$. We in general define the K-adic equation

$$c_1\mathbf{a}_1{}^K + c_2\mathbf{a}_2{}^K + \cdots + c_n\mathbf{a}_n{}^K = 0 \tag{15}$$

to be equivalent to the system of n equations

$$\sum_h [c_h a_{hi} a_{hj} \cdots \text{to } K \text{ factors}] = 0 \tag{16}$$

where in each equation h runs from 1 to n. Passage from one equation of the system to another is by varying each of the second subscripts from 1 to N. By inspection of (4) it is evident that the eliminant of the set of n equations is the same as C_0. We thus have

Theorem II. *A set of* n *symmetrical K-ads are linearly related if and only if their vector elements lie on a hypercone of order* K.

4. Special Forms of the General Identity: Method of Standard Sets.

By assigning particular values to the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$ and by selecting particular forms of polynomials $F(\mathbf{a})$, a great number of

identities may be obtained from the scalar formula (5) as well as from the vector formula (9).

An elementary method is, evidently, to assign numerical values to the elements a_{ij}, with the sole restriction that C_0 shall not vanish. The coefficients $C_1, C_2, \cdots C_n$ are polynomials of degree K. By (5), an arbitrary polynomial $F(x)$ can be expanded in terms of these n polynomials, the coefficients $F(\mathbf{a}_i)$ being found by direct substitution.

It is evident, therefore, that $C_1, C_2, \cdots C_n$ are linearly independent. It is equally evident that not every linearly independent set of n polynomials homogeneous of degree K in N variables can be taken as these n C's; for the totality of all vectors where two or more C's vanish is comprised by the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$. We may embody this distinction in the following definition:

Definition I. *A set of polynomials* $P_1, P_2, \cdots P_n$, *homogeneous of degree* K *in* N *variables, such that* $P_i(\mathbf{a}_j)$ *vanishes when* i *is different from* j *but not when* i *equals* j *for each of the points* $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$, *will be called a* **standard set.**

By use of standard sets of polynomials, many of the identities of elementary algebra may be made to appear as special cases of (5). To take the simplest of illustrations, let $N = K = 2$ so that $n = 3$, with variables x, y. Let the matrix a_{ij} be

$$\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}$$

whence the determinant (4) becomes

$$\begin{vmatrix} F(x, y), & x_2, & xy, & y_2 \\ F(1, 0), & 1\,, & 0, & 0 \\ F(1, 1), & 1\,, & 1, & 1 \\ F(0, 1), & 0\,, & 0, & 1 \end{vmatrix}$$

By easy calculation $C_0 = 1, C_1 = xy - x^2, C_2 = - xy, C_3 = xy - y^2$. Choosing $F(x, y) = (x + y)\,(x - y)$ yields $F(1, 0) = 1$, $F(1, 1) = 0$, $F(0, 1) = -1$. The identity (5) appears as

$$(x+y)\,(x-y) + (xy - x^2) + 0 + (-1)\,(xy - y^2) = 0$$

To take a less familiar illustration, let $\mathbf{F}(\mathbf{x})$ be a quadratic vector function in ordinary space, $N=3$, $K=2$. A simple way to build a standard set of polynomials is to choose four vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ of which no three are coplanar, and form six products of linear factors $\mathbf{x}\cdot\mathbf{b}_i\mathbf{b}_j\cdot\mathbf{x}$ which are the required polynomials. The six fixed points or

vectors $\mathbf{a}_h$ are the six intersections of four planes, and the vector identity (9) becomes, by putting $\mathbf{F}(\mathbf{a}_h) = \mathbf{F}(\mathbf{b}_i \times \mathbf{b}_j) = \mathbf{f}_{pq}$,

$$\mathbf{F}(\mathbf{x}) = \mathbf{x}\cdot(\mathbf{b}_1\mathbf{f}_{23}\mathbf{b}_4 + \mathbf{b}_1\mathbf{f}_{34}\mathbf{b}_2 + \mathbf{b}_1\mathbf{f}_{24}\mathbf{b}_3 + \mathbf{b}_2\mathbf{f}_{14}\mathbf{b}_3 + \mathbf{b}_2\mathbf{f}_{13}\mathbf{b}_4 + \mathbf{b}_3\mathbf{f}_{12}\mathbf{b}_4)\cdot\mathbf{x} \quad (17)$$

where, if $\mathbf{F}(\mathbf{x})$ is an arbitrary quadratic vector function, the six vectors $\mathbf{f}$ are arbitrary.

In both the above illustrations we have $K = 2$, and note that the polynomials of the standard set are formed by selecting pairs of linear polynomial from a group of $K + N - 1$ possibilities. This procedure is applicable in all cases. For the number of ways in which K factors can be selected from a group of $K + N - 1$ factors with no repetitions is the same as the number of ways in which K variables can be selected from N variables allowing repetitions, that is n ways by definition of n. Each linear factor is of the form $\mathbf{b}_i\cdot\mathbf{x}$ where

$$\mathbf{b}_i\cdot\mathbf{x} = b_{i1}x_1 + b_{i2}x_2 + \cdots + b_{iN}x_N \quad (18)$$

and there are $N + K - 1$ vectors $\mathbf{b}_i$ so chosen that no N of them are linearly related. We thus arrive at the following:

Definition II. *A standard set of polynomials which has been formed by taking products of* K *linear polynomials selected from* $N + K - 1$ *such polynomials without repetition will be called a* **factored set**.

The vectors $\mathbf{a}_1, \mathbf{a}_2 \cdots \mathbf{a}_n$ may always be chosen so that $P_i(\mathbf{a}) = 1$. We define the vector $[\mathbf{b}_p\mathbf{b}_q\mathbf{b}_r\cdots$ to $N - 1$ factors$]$ to be the vector whose scalar components are the respective cofactors of the elements of the first row from the determinant

$$\begin{vmatrix} b_{i1}, & b_{i2}, & b_{i3}, & \cdots, & b_{iN} \\ b_{p1}, & b_{p2}, & b_{p3}, & \cdots, & b_{pN} \\ b_{q1}, & b_{q2}, & b_{q3}, & \cdots, & b_{qN} \\ b_{r1}, & b_{r2}, & b_{r3}, & \cdots, & b_{rN} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \text{to } N \text{ rows} \end{vmatrix} \quad (19)$$

We then define one of the vectors $\mathbf{a}$, say $\mathbf{a}_1$, as follows; let $\mathbf{m}_1$ denote the vector $[\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\cdots\mathbf{b}_{N-1}]$ and write

$$\mathbf{a}_1 = \frac{\mathbf{m}_1}{(\mathbf{b}_N\cdot\mathbf{m}_1\mathbf{b}_{N+1}\cdot\mathbf{m}_1\cdots\mathbf{b}_{N+K-1}\cdot\mathbf{m}_1)^{\frac{1}{K}}} \quad (20)$$

The other vectors $\mathbf{a}$ are built up in a similar manner: each $\mathbf{a}$ is a function of all the $\mathbf{b}$'s; the denominator is the Kth root of the product

of K determinants of the form $\mathbf{b}\cdot\mathbf{m}$, that is of the form (19); any selection of $N - 1$ of the $\mathbf{b}$'s determines one of the n vectors $\mathbf{m}_1$, $\mathbf{m}_2 \cdots \mathbf{m}_n$ which forms the numerator of the corresponding $\mathbf{a}$. All the remaining $\mathbf{b}$'s occur explicitly in the denominator.

The polynomials P_1, $P_2, \cdots P_n$ may now be taken of the form

$$P_h = (\mathbf{b}_p\cdot\mathbf{x})\,(\mathbf{b}_q\cdot\mathbf{x})\,(\mathbf{b}_r\cdot\mathbf{x})\cdots \text{to } K \text{ factors.} \tag{21}$$

The $\mathbf{a}$ of corresponding subscript is that $\mathbf{a}$ which contains precisely the same choice of $\mathbf{b}$'s in its denominator; for example, if $\mathbf{a}_1$ be defined by (20) then P_1 must be taken as

$$P_1 = \mathbf{b}_N\cdot\mathbf{x}\mathbf{b}_{N+1}\cdot\mathbf{x}\cdots\mathbf{b}_{N+K-1}\cdot\mathbf{x} \tag{22}$$

By inspection of (20) and (22) it is evident that $P_1(\mathbf{a}_1) = 1$, and similarly for any other distribution of $\mathbf{b}$'s between numerator and, denominator. That is $P_h(\mathbf{a}_h) = 1$ for all values of h, as was to be shown.

It is clear that a set of polynomials defined by (21) will fulfill the conditions for a standard set provided no group of N vectors chosen from the $\mathbf{b}$'s are linearly related. For the $\mathbf{a}$'s may be taken as in (20) and no denominator can vanish. At least one of the $\mathbf{b}$'s which occur in $\mathbf{m}_j$ must occur in P_i when i is different from j. Hence $P_i(\mathbf{a}_j) = 0$.

Definition III. *A factored set of polynomials, together with a set of points* $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$ *such that* $P_i(\mathbf{a}_j)$ *vanishes when* i *is different from* j *but equals unity when* i *equals* j, *will be called a* **normal reference system**.

Any polynomial P_i of a normal reference system agrees with the coefficient C_i of (5) except for a constant factor, since P_i and C_i both vanish at every $\mathbf{a}$ except $\mathbf{a}_i$. By putting $\mathbf{a}_i$ for $\mathbf{x}$ in (5) we have

$$F(\mathbf{a}_1)C_0 + F(\mathbf{a}_i)C_i(\mathbf{a}_i) = 0; \tag{23}$$

but F is an arbitrary polynomial, hence $F(\mathbf{a}_i)$ is in general not zero. Therefore we have identically

$$C_0 + C_i(\mathbf{a}_i) = 0 \tag{24}$$

and since $P_i(\mathbf{a}_i) = 1$ it follows that

$$P_i = -\frac{C_i}{C_0} \tag{25}$$

By substituting $C_i = -C_0P_i$ in the vector identity (9) we arrive at the following special case of theorem I,—

Theorem III. *An arbitrary vector function homogeneous of degree* K *in* N *variables may be written as the sum of* n *terms*

$$\mathbf{F}(\mathbf{x}) = \sum_h [\mathbf{f}_h(\mathbf{b}_p \cdot \mathbf{x})\,(\mathbf{b}_q \cdot \mathbf{x}) \cdots \text{to } K \text{ factors}] \tag{26}$$

where the vectors $\mathbf{f}_1, \mathbf{f}_2, \cdots \mathbf{f}_n$ *are arbitrary; in each term occur* K *linear polynomials selected from* N + K − 1 *linear polynomials; these linear polynomials are under the sole restriction that no* N *of them are linearly related.*

It follows from the form of (26) that, if $\mathbf{a}_h$ be of the form (20),

$$\mathbf{F}(\mathbf{a}_h) = \mathbf{f}_h \tag{27}$$

It is apparent that expansions like (26) will be of advantage in investigations where symmetry of form is to be sought. As a simple illustration suppose $K = 2$. We have $N + K - 1 = N + 1$ and may take as our linear polynomials the N variables $x_1, x_2, \cdots x_n$ together with their sum $x_i + x_2 + \cdots + x_N$. We see at once that any quadratic vector function can be written in the form

$$\Sigma \mathbf{f}_{ij} x_i x_j + \varphi(\mathbf{x}) \Sigma x_h; \;\; [i \text{ different from } j;\; h = 1, 2, \cdots N] \tag{28}$$

where $\varphi(\mathbf{x})$ is a linear vector function in N-space.

5. Special Forms of the General Identity: Method of Factoring.

I propose next to consider a large class of identities which we may obtain from (5) and from (9) if, instead of making some special choice of the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$, we take particular polynomials $F(\mathbf{x})$ and $\mathbf{F}(\mathbf{x})$.

As a first example, let $E(\mathbf{x})$ be a polynomial of degree $K - 1$ and take $\mathbf{F}(\mathbf{x}) = \mathbf{x}E(\mathbf{x})$; that is, the vector function $\mathbf{F}(\mathbf{x})$ is chosen to be the point vector $\mathbf{x}$ multiplied by an arbitrary scalar polynomial. The identity (9) becomes

$$\mathbf{x}E(\mathbf{x})C_0 + \mathbf{a}_1 E(\mathbf{a}_1)C_1 + \mathbf{a}_2 E(\mathbf{a}_2)C_2 + \cdots + \mathbf{a}_n E(\mathbf{a}_n)C_n = 0 \tag{29}$$

We may now choose the polynomial $E(\mathbf{x})$ so that it shall vanish at a number of points, say at m points. Let these points be $\mathbf{a}_{n-m+1}$, $\mathbf{a}_{n-m+2}$, etc. up to $\mathbf{a}_n$. In other words let $E(\mathbf{x})$ be a function of the same type as the C's but associated with the case $K - 1$. This particular form of $E(\mathbf{x})$ may be distinguished as $E_m(\mathbf{x})$. The last m terms of (29) will now vanish and we have

$$\mathbf{x}E_m(\mathbf{x})C_0 + \mathbf{a}_1E_m(\mathbf{a}_1)C_1 + \cdots + \mathbf{a}_{n-m}E_m(\mathbf{a}_{n-m})C_{n-m} = 0 \tag{30}$$

If the number of terms in the general polynomial of degree K in N variables be denoted by $n(N, K)$ we have the identity

$$n(N, K) - n(N, K-1) = n(N-1, K) \tag{31}$$

while $m = n(N, K-1) - 1$. The number of terms in (29) is $n(N, K) + 1$. Hence the number of terms in (30) is $n(N-1, K) + 2$. It is clear that we may obtain from (29) as many identities of the form (30) as there are ways of selecting m points from $n + 1$ points.

Again, we may take $F(\mathbf{x}) = L(\mathbf{x})E(\mathbf{x})$ where $L(\mathbf{x})$ is a linear polynomial. Repeating the same steps, with the same meaning of $E_m(\mathbf{x})$, we shall obtain, instead of the vector identity (30), a scalar identity

$$L(\mathbf{x})E_m(\mathbf{x})C_0 + L(\mathbf{a}_1)E_m(\mathbf{a}_1)C_1 + \cdots + L(\mathbf{a}_{n-m})E_m(\mathbf{a}_{n-m})C_{n-m} = 0 \tag{32}$$

We may now choose $L(\mathbf{x})$ so as to vanish at the $N - 1$ points whose subscripts are $n - m - N + 2$, $n - m - N + 3$, etc. up to $n - m$; that is, $L(\mathbf{x})$ may be taken to be the determinant of the components of these $N - 1$ point vectors and the variable vector $\mathbf{x}$. The last $N - 1$ terms of (32) will now vanish, leaving $n(N-1, K) - N + 3$ terms. We may obtain as many identities of this last form as there are ways of selecting $N - 1$ points from $n(N-1, K) + 2$ points, multiplied by the number of ways (above noted) in which (30) could be formed.

Instead of factoring $F(\mathbf{x})$ into a linear polynomial and one of degree $K - 1$ we can evidently obtain other identities in a similar manner by factoring $F(\mathbf{x})$ into any pair of polynomials, or into any number of polynomials such that the sum of their degrees is equal to K. Clearly then there exist a vast number of identities connecting the coefficients C in (5) or (9) with the anologous coefficients in expansions associated with smaller values of K.

In a former paper [1] I have made a somewhat detailed study of the case $K = 2$, $N = 3$. The "Aconic Function" of Hamilton is a function of six vectors, in the form of a scalar product, which vanishes when these six vectors lie on a quadric cone; equivalent, therefore, to C_0 for the case in question. In the paper referred to, the properties of the seven C's for this case were derived from Hamilton's function,

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1 An Identical Relation Connecting Seven Vectors. Proc. Royal Soc. Edin. vol. XL, Part II(No. 14), pp. 129–139.

instead of from the determinant (4). It was shown that in this case any three C's are connected by a relation of the form

$$L_pC_p + L_qC_q + L_rC_r = 0 \tag{33}$$

where the L's are linear polynomials. It is easy to see from (30) that a similar relation holds for all values of N and K. For, writing for the sake of brevity

$$n - m - N + 1 = n(N-1, K) - N + 2 = q; \; m = n(N, K-1) - 1 \tag{34_1}$$

and with a notation like that of the preceding article,

$$\mathbf{b} = [\mathbf{x}\mathbf{a}_{q+2}\mathbf{a}_{q+3}\cdots\mathbf{a}_{n-m}] \tag{34_2}$$

we multiply $\mathbf{b}$ into (30), taking the dot product. The first term vanishes because $\mathbf{b}\cdot\mathbf{x} = 0$. The last $N - 2$ terms also vanish. The products $\mathbf{b}\cdot\mathbf{a}_1$ etc. up to $\mathbf{b}\cdot\mathbf{a}_{q+1}$ are obviously linear polynomials in $\mathbf{x}$, while $E_m(\mathbf{a}_1)$ etc. are constants. We now have

$$\mathbf{b}\cdot\mathbf{a}_1E_m(\mathbf{a}_1)C_1 + \cdots + \mathbf{b}\cdot\mathbf{a}_{q+1}E_m(\mathbf{a}_{q+1})C_{q+1} = 0 \tag{35}$$

A standard set of polynomials, as already pointed out, becomes a set of C's when each is multiplied by a proper constant. We have therefore by (35) the following theorem:

Theorem IV. *Any $q + 1$ polynomials selected from a standard set are connected by relations of the form $\Sigma \mathrm{L}_i\mathrm{P}_i = 0$, where $\mathrm{L}_1\cdots\mathrm{L}_{q+1}$ are linear polynomials of the form $\mathbf{b}\cdot\mathbf{a}_i$.*

If $S[p, q]$ be the number of ways in which p points may be selected from q points, (i.e. S is a binomial coefficient), the number of such relations connecting the polynomials of a given standard set is $S[q + 1, n]S[N - 2, n - q - 1]$.

The factor last written may be denoted by Q, that is

$$Q = S[N - 2, n - q - 1] \tag{36}$$

This is the number of relations of the form $\Sigma L_iP_i = 0$ which may be written down connecting a *given group* of $q + 1$ polynomials selected from the n polynomials of a standard set. (We assume $N > 2, K > 1$). If $N = 3$, $K = 2$, we have $n = 6$, $q = 2$, $Q = 3$. If either N or K be larger, we have $Q > q + 1$.

We may now prove that, regarding a relations of form (35) as equations satisfied by a particular selection of $q + 1$ of the C's, not more than q of these equations can be independent. For consider the $m + N - 2$ points $\mathbf{a}_{q+2}$, $\mathbf{a}_{q+3}, \cdots \mathbf{a}_n$. These are the $\mathbf{a}$'s which

correspond to the terms absent from (35). Let $E_1, E_2, \cdots E_{q+1}$ be polynomials of degree $K - 1$ which vanish, respectively, at $q + 1$ different selected groups of m points chosen from the above $m + N - 2$ points. Let $\mathbf{b}_1, \mathbf{b}_2, \cdots \mathbf{b}_{q+1}$ denote those linear vector functions of $\mathbf{x}$ of form (34) into which, in each case, enter the $\mathbf{a}$'s not occurring in the corresponding E. Corresponding to each choice of E and corresponding $\mathbf{b}$ we may write an identity of the form (35). Thus

$$\begin{aligned}
&\mathbf{b}_1\cdot\mathbf{a}_1E_1(\mathbf{a}_1)C_1 + \cdots + \mathbf{b}_1\cdot\mathbf{a}_{q+1}E_1(\mathbf{a}_{q+1})C_{q+1} = 0\\
&\mathbf{b}_2\cdot\mathbf{a}_1E_2(\mathbf{a}_1)C_1 + \cdots + \mathbf{b}_2\cdot\mathbf{a}_{q+1}E_2(\mathbf{a}_{q+1})C_{q+1} = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\mathbf{b}_{q+1}\cdot\mathbf{a}_1E_{q+1}(\mathbf{a}_1)C_1 + \cdots + \mathbf{b}_{q+1}\cdot\mathbf{a}_{q+1}E_{q+1}(\mathbf{a}_{q+1})C_{q+1} = 0
\end{aligned} \tag{37}$$

We may regard these as $q + 1$ linear equations satisfied by the $q + 1$ C's which enter. We know that these C's are actual polynomials of degree K in $\mathbf{x}$. Hence the equations cannot be independent, as was to be demonstrated.

On the other hand we may in general select q such equations which shall be independent. For consider the coefficients of C_j in the successive equations: they are of the form $\mathbf{b}_i\cdot\mathbf{a}_jE_i(\mathbf{a}_j)$, and may be thought of as polynomials of degree K in $\mathbf{a}_j$. They all vanish if $\mathbf{a}_j$ be made to coincide with any one of the points $\mathbf{a}_{q+2}\cdots\mathbf{a}_n$ or $\mathbf{x}$, by hypothesis, that is at $m + N - 1$ points. In general n polynomials may be linearly independent. Hence of these polynomials $n - (m + N - 1)$ may be linearly independent. But this number is q, as was to be shown. A specific rule for selecting the q equations will be given below.

Corollary to Theorem IV. The following corollary to theorem IV will be essential in the applications made below. We suppose given a standard set of polynomials $P_1(\mathbf{x}),\cdots P_n(\mathbf{x})$ based on a set of vectors $\mathbf{a}_1,\cdots\mathbf{a}_n$ as already defined. We now adjoin $N - 3$ other vectors which need not be distinct from the others, and which we may call $\mathbf{a}_{n+1}\cdots$etc.; and write as in (34_2)

$$\mathbf{b}_1 = [\mathbf{x}\mathbf{a}_h\mathbf{a}_j\cdots] \tag{37_A}$$

where the $N - 2$ vectors $\mathbf{a}_h$, $\mathbf{a}_j,\cdots$ which occur in $\mathbf{b}_i$ are precisely those $\mathbf{a}$'s (out of the total number of $n + N - 3$), which do not occur in P_i. It is evident that $\mathbf{b}_i\cdot\mathbf{y}$ will be a linear polynomial in $\mathbf{x}$. Write

$$L_i(\mathbf{x}) = \mathbf{b}_i\cdot\mathbf{y} \tag{37_B}$$

where $\mathbf{y}$ is an arbitrary vector. The corollary may now be stated:

Of the polynomials of a standard set, not fewer than $n - m - t$ can be connected by a relation of the form

$$\Sigma c_i L_i(\mathbf{x}) P_i(\mathbf{x}) = 0 \tag{37$_C$}$$

unless the linear polynomials $L_i(\mathbf{x})$ *all vanish at more than* t *of the points* $\mathbf{a}_1, \cdots \mathbf{a}_n$; *it is understood that* $L_1(\mathbf{x})$ *is of the form* (37$_B$), *and that* c_i *is independent of* $\mathbf{x}$.

Proof. Suppose a relation of the form

$$c_1 L_1(\mathbf{x}) P_1(\mathbf{x}) + \cdots + c_w L_w(\mathbf{x}) P_w(\mathbf{x}) = 0 \tag{37$_D$}$$

where $w = n - m - t - 1$ and where the L's all vanish at the t points $\mathbf{a}_{w+1}, \cdots, \mathbf{a}_{w+t}$, but at no others of subscript less than $n + 1$.

The polynomial $L_i(\mathbf{x})$ is the determinant of the coefficients of the N vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{a}_i$, and the $N - 3$ adjoined vectors, any or all of which might coincide with an equal number of the original set $\mathbf{a}_1 \cdots \mathbf{a}_n$, according to the value of t. Thus $L_i(\mathbf{x})$ is a linear function of $\mathbf{a}_i$, and $L_j(\mathbf{x})$ is the same linear function of $\mathbf{a}_j$ except perhaps in sign. If we write $L_i(\mathbf{x}) = L(\mathbf{a}_i)$ we may write (37$_D$) in the form

$$c_1 L(\mathbf{a}_1) P_1(\mathbf{x}) + \cdots + c_w L(\mathbf{a}_w) P_w(\mathbf{x}) = 0 \tag{37$_E$}$$

It suffices to show that $L(\mathbf{a}_{n-m}) = 0$ contrary to hypothesis.

Now c_i must be a polynomial of degree $K - 1$ in $\mathbf{a}_i$. For all the terms of the supposed identity (37$_E$) are of degree K in $\mathbf{a}_i$ with exception of the ith term; is independent of $\mathbf{a}_i$; and $L(\mathbf{a}_i)$ is linear in $\mathbf{a}_i$.

Furthermore c_i is of degree $K - 1$ in each of the $\mathbf{a}$'s from $\mathbf{a}_{n-m+1}$ to $\mathbf{a}_n$ inclusive. For (37$_E$), being an identity, will subsist if we make $\mathbf{a}_i$ coincide with $\mathbf{a}_{n-m+j}$; P_i will be unaltered since it is independent of $\mathbf{a}_i$; all other P's vanish; $L(\mathbf{a}_{n-m+j})$ by hypothesis does not vanish; hence c_i must vanish when $\mathbf{a}_i$ coincides with $\mathbf{a}_{n-m+j}$; that is, c_i is a polynomial of degree $K - 1$ in $\mathbf{a}_i$ vanishing at the $m = n(N, \mathrm{K} - 1) - 1$ points $\mathbf{a}_{n-m+1} \cdots \mathbf{a}_n$, and is therefore a polynomial of degree $K - 1$ in each of these points. We may accordingly identify the c's of (37$_E$) with the polynomials $E_m(\mathbf{a}_i)$ of (32) except, in each case, for a constant multiplier g_i.

If, finally, we make $\mathbf{a}_i$ coincide with $\mathbf{a}_{n-m}$, c_i will not vanish, for a polynomial of degree $K - 1$ cannot be made to vanish at more than m arbitrary points; P_i is independent of $\mathbf{a}_i$; and so $L(\mathbf{a}_{n-m})$ must vanish, contrary to hypothesis.

Hence the identity (37$_D$) is impossible unless the L's all vanish at more than t of the points $\mathbf{a}_1 \cdots \mathbf{a}_n$, as was to be proved.

6. Application to the Problem of Expressing Polynomials as Determinants whose Elements are Linear Polynomials.

If we take $M = 3$ we have $q = n(2, K) - 1 = K$. We may then regard K equations of type (37) as so many linear equations in the $K + 1$ unknowns $C_1, \cdots C_{K+1}$. The K-row determinants from the matrix of the coefficients are polynomials of degree K; their elements are linear in $\mathbf{x}$; and, aside from a constant factor, they must be equal to the respective C's which enter. We therefore have the means of setting up polynomials of degree K in 3 variables in the form of determinants with linear elements and vanishing at $n - 1$ given points.

By use of the theory of matrices, Dickson has recently given a highly elegant proof that all homogeneous polynomials in two and in three variables, as well as quadrics and sufficiently general cubics in four variables, can be expressed as determinants with linear elements; and that no other general polynomials can be so expressed.[2] He has shown how such a transformation can be accomplished for an arbitrary plane curve by employing no irrationalities except the roots of a single equation of degree K. The present investigation is equally general if we have no regard for rationality: a polynomial of degree K through $n - 1$ arbitrary points is an arbitrary polynomial. With regard to rationality, however, the method of the present article is less general, for Dickson does not assume any points on the curve to be known. But an explicit formulation of the polynomial as a function of the n points $\mathbf{a}_1, \cdots \mathbf{a}_{n-1}$ and $\mathbf{x}$ is not without utility.

To take a concrete case, let $N = 3$ and $K = 3$. Let it be required to express as a determinant with linear elements the cubic through the nine points $\mathbf{a}_1 \cdots \mathbf{a}_9$. Let the six points $\mathbf{a}_1 \cdots \mathbf{a}_6$ be regarded as $n(3, 2)$ points corresponding to the case $N = 3$, $K = 2$. Let $E_1 \cdots E_6$ be the respective quadrics which vanish at five of these six points, the omitted point having corresponding subscript. Let (xij) and (kij) denote the determinants of the components of the vectors $\mathbf{x}$, $\mathbf{a}_i$, $\mathbf{a}_j$ and $\mathbf{a}_k$, $\mathbf{a}_i$, $\mathbf{a}_j$ respectively. We may chose $\mathbf{b}_1 = [\mathbf{x}\mathbf{a}_i]$; so that $\mathbf{b}_i \cdot \mathbf{a}_j = (xij)$. We may write $\mathbf{a}_{10}$ for an arbitrary tenth point. With the notation already used C_{10} will be our required cubic. We may form six equations of the form (37) each of which involves the cubics $C_7 \cdots C_{10}$. Three of these are sufficient and may be taken thus:

$$(x17)E_1(\mathbf{a}_7)C_7 + (x18)E_1(\mathbf{a}_8)C_8 + (x19)E_1(\mathbf{a}_9)C_9 + (x1_{10})E_1(\mathbf{a}_{10})C_{10} = 0$$
$$(x27)E_2(\mathbf{a}_7)C_7 + (x28)E_2(\mathbf{a}_8)C_8 + (x29)E_2(\mathbf{a}_9)C_9 + (x2_{10})E_2(\mathbf{a}_{10})C_{10} = 0$$
$$(x37)E_3(\mathbf{a}_7)C_7 + (x38)E_3(\mathbf{a}_8)C_8 + (x39)E_3(\mathbf{a}_9)C_9 + (x3_{10})E_3(\mathbf{a}_{10})C_{10} = 0$$

[2] Trans. Amer. Math. Soc. Vol. 22, No. 2 (April 1921), pp. 167–179.

Hence we may take as the required cubic the determinant

$$\begin{vmatrix} (x17)E_1(\mathbf{a}_7)\,, & (x18)E_1(\mathbf{a}_8)\,, & (x19)E_1(\mathbf{a}_9) \\ (x27)E_2(\mathbf{a}_7)\,, & (x28)E_2(\mathbf{a}_8)\,, & (x29)E_2(\mathbf{a}_9) \\ (x37)E_3(\mathbf{a}_7)\,, & (x38)E_3(\mathbf{a}_8), & (x39)E_3(\mathbf{a}_9) \end{vmatrix} \tag{38}$$

That this determinant vanishes when $\mathbf{x}$ coincides with $\mathbf{a}_1$ is evident, for the elements of the first row all vanish; similarly the second and third rows vanish when $\mathbf{x}$ coincides with $\mathbf{a}_2$ and $\mathbf{a}_3$ respectively. The columns vanish when $\mathbf{x}$ coincides with $\mathbf{a}_7$, $\mathbf{a}_8$, $\mathbf{a}_9$. Since as has been shown the determinant must also vanish when $\mathbf{x}$ coincides with $\mathbf{a}_4$, $\mathbf{a}_5$, or $\mathbf{a}_6$, we are presented with three new identities of the form

$$\begin{vmatrix} (417)E_1(\mathbf{a}_7), & (418)E_1(\mathbf{a}_8), & (419)E_1(\mathbf{a}_9) \\ (427)E_2(\mathbf{a}_7), & (428)E_2(\mathbf{a}_8), & (429)E_2(\mathbf{a}_9) \\ (437)E_3(\mathbf{a}_7), & (438)E_3(\mathbf{a}_8), & (439)E_3(\mathbf{a}_9) \end{vmatrix} = 0 \tag{39}$$

the other two having 5 and 6 in place of 4. In fact if we multiply the elements of the first row of (39) by the three-row determinant (561), those of the second row by (562), of the third by (563), and add, the sum of the elements of each column is zero; for

$$(561)\,(41x)E_1(\mathbf{x}) + (562)\,(42x)E_2(\mathbf{x}) + (563)\,(43x)E_3(\mathbf{x}) = 0 \tag{40}$$

is an identity of the type (33), a special case of (35). It holds therefore when $\mathbf{x}$ is replaced by $\mathbf{a}_7$, $\mathbf{a}_8$, or $\mathbf{a}_9$.

Determinants of similar form to (38) may evidently be written down at once for any value of K when $N = 3$. The E's will in all cases denote polynomials of degree $K - 1$; vanishing at $n(N, K - 1) - 1$ points.

When N is greater than 3 we have always $q > K$; hence if we select from the equations of form (37) a set of q equations which are independent we may write the C's proportional to a set of determinants having linear elements; *these determinants must accordingly be reducible polynomials and must possess a common factor of degree* q − K, *the other factor being one of the* C*'s.*

Not every set of q equations of the type of (37) will be independent. For example let $N = 4$, $K = 2$, so that $n = 10$, $m = 3$, $q = 4$, and $Q = S[2, 5] = 10$. If we choose our four vectors $\mathbf{b}_1 \cdots \mathbf{b}_4$ to be $[\mathbf{x}\mathbf{a}_i\mathbf{a}_5]$ where $i = 1, 2, 3, 4$, the coefficients of C_j in our four equations will be $(xi5j)$ $(pqrj)$ where p, q, and r are different from i and from each other and are less than five; while j has any value from six to ten inclusive. The four polynomials in $\mathbf{a}_j$ are linearly related; for we may write as a special case of (29)

$$\mathbf{a}_j(1234) - \mathbf{a}_1(j234) + \mathbf{a}_2(j134) - \mathbf{a}_3(j124) + \mathbf{a}_4(j123) = 0 \quad (41)$$

and by multiplying by $[\mathbf{x}\mathbf{a}_5\mathbf{a}_j]$

$$(x5j1)\,(j234) - (x5j2)\,(j134) + (x5j3)\,(j124) - (x5j4)\,(j123) = 0 \quad (42)$$

which, being an identity, holds when j runs from 6 to 10. Hence when $\mathbf{a}_5$ is common to all four $\mathbf{b}$'s the four equations are linearly related.

When the four $\mathbf{b}$'s do not contain an $\mathbf{a}$ in common the four equations are independent. For suppose the $\mathbf{b}$'s to be $[\mathbf{x}\mathbf{a}_1\mathbf{a}_5]$, $[\mathbf{x}\mathbf{a}_2\mathbf{a}_5]$, $[\mathbf{x}\mathbf{a}_3\mathbf{a}_5]$, and $[\mathbf{x}\mathbf{a}_1\mathbf{a}_2]$. The four corresponding equations are $\Sigma(x15j)(234j)C_j = 0$ where j runs from 6 to 10, and three other equations of similar form. The determinant of the coefficients of the first four columns is

$$\begin{vmatrix} (x156)\,(2346), & (x157)\,(2347), & (x158)\,(2348), & (x159)\,(2349) \\ (x256)\,(1346), & (x257)\,(1347), & (x258)\,(1348), & (x259)\,(1349) \\ (x356)\,(1246), & (x357)\,(1247), & (x358)\,(1248), & (x359)\,(1249) \\ (x126)\,(3456), & (x127)\,(3457), & (x128)\,(3458), & (x129)\,(3459) \end{vmatrix} \quad (43)$$

In order that this determinant might vanish identically it would be necessary and sufficient that four numbers c_1, c_2, c_3, c_4 exist, independent of $\mathbf{a}_j$ and not all zero, such that

$$c_1(x15j)\,(234j) + c_2(x25j)\,(134j) + c_3(x35j)\,(124j) + c_4(x12j)\,(345j) = 0 \quad (44)$$

for the vectors $\mathbf{a}_6$, $\mathbf{a}_7$, $\mathbf{a}_8$, $\mathbf{a}_9$, represented by $\mathbf{a}_j$, are arbitrary. If we let $\mathbf{a}_j = \mathbf{x} + \mathbf{a}_5$ this equation reduced to its last term, namely

$$c_4(x125)\,(345x) = 0$$

whence $c_4 = 0$, for the vectors $\mathbf{x}$, $\mathbf{a}_1 \cdots \mathbf{a}_5$ may have any values whatever. By letting $\mathbf{a}_j$ be $\mathbf{a}_1 + \mathbf{a}_4$, $\mathbf{a}_2 + \mathbf{a}_4$, and $\mathbf{a}_3 + \mathbf{a}_4$ we see that c_1, c_2, and c_3 must all vanish, contrary to hypothesis. Hence (43) does not vanish identically. We may proceed similarly whenever the four $\mathbf{b}$'s do not have an $\mathbf{a}$ in common.

This determinant must accordingly contain as a factor C_{10}, a quadric through the nine points $\mathbf{a}_1, \cdots \mathbf{a}_9$. That it vanishes at all these points can be verified by inspection: if $\mathbf{x} = \mathbf{a}_4$ the last two rows are equal numerically and opposite in sign; if $\mathbf{x}$ equals any other $\mathbf{a}$ all the elements of a row or of a column vanish.

This determinant is of the fourth degree in $\mathbf{x}$ and in $\mathbf{a}_1 \cdots \mathbf{a}_5$. It is of the second degree in $\mathbf{a}_6 \cdots \mathbf{a}_9$. Since the factor C_{10} is a quadric in

all ten vectors the other factor must be a quadric in $\mathbf{x}$ and in $\mathbf{a}_1 \cdots \mathbf{a}_5$ and be independent of $\mathbf{a}_6 \cdots \mathbf{a}_9$. If we wish to find this latter factor it is allowable to assign to $\mathbf{a}_6 \cdots \mathbf{a}_9$ any values we please. If we take

$$\mathbf{a}_6 = \mathbf{a}_3 + \mathbf{a}_4, \quad \mathbf{a}_7 = \mathbf{a}_2 + \mathbf{a}_4, \quad \text{and } \mathbf{a}_8 = \mathbf{a}_2 + \mathbf{a}_3 \tag{45}$$

the determinant (43) becomes

$$\begin{vmatrix} 0 , & 0 , & 0 , & (x159)(2349) \\ 0 , & (x254)(1342), & (x253)(1342), & (x259)(1349) \\ (x354)(1243), & 0 , & (x352)(1243), & (x359)(1249) \\ 0 , & (x124)(3452), & (x123)(3452), & (x129)(3459) \end{vmatrix} \tag{46}$$

which is equal to the product of $(x354)\,(1243)\,(x159)\,(2349)$ times the determinant

$$\begin{vmatrix} (x254)(1342), & (x253)(1342) \\ (x124)(3452), & (x123)(3452) \end{vmatrix} .$$

which in turn is

$$(1342)\,(3452)\,[(x254)\,(x123) - (x124)\,(x253)]; \tag{48}$$

but from (41), by letting $\mathbf{a}_i = \mathbf{x}$ and multiplying by $[\mathbf{x}\mathbf{a}_2\mathbf{a}_5]$

$$-(x251)\,(x234) - (x253)\,(x124) + (x254)\,(x123) = 0 \tag{49}$$

that is, the quantity in brackets in (48) is equal to $(x251)\,(x234)$. Collecting results, we see that the determinant (43), by virtue of the substitutions (45), become the product of eight determinants:

$$(x354)\,(x251)\,(x234)\,(x159)\,(1243)\,(1342)\,(3452)\,(2349) \tag{50}$$

It is evident by inspection of this product that the adventitious factor of the determinant (43) must be

$$(x354)\,(x251)\,(1234) \tag{51}$$

for in no other way can we pick from (50) a factor quadratic in $\mathbf{a}_1 \cdots \mathbf{a}_5$ and $\mathbf{x}$. The algebraic sign is, however, undetermined.

Determinants similar in form to (43) can evidently be written down for any values of K and N. The second factor of each element will in every case be a polynomial of degree $K - 1$ in the variables which enter into it. The first factor is linear in $\mathbf{x}$ and in $N - 1$ other points. We have therefore the theorem:

Theorem V. *Given a polynomial homogeneous of degree* K *in* N *variables, and* n − 1 *points at which this polynomial vanishes: it is in general possible to write down a determinant of order* q *whose elements*

are linear in $\mathbf{x}$ *and are rational functions of these points and which shall contain the polynomial as a factor.*

One further illustration may be given of selecting the $\mathbf{b}$'s so that the determinant shall not vanish identically, and of factoring the determinant. Take $N = 5$, $K = 2$, and let the seven $\mathbf{b}$'s be $[x145]$, $[x356]$, $[x346]$, $[x567]$, $[x467]$, $[x457]$, and $[x456]$. The elements of the first row of our determinant will be $(x145j)\ (2367j)$ where $j = 8, 9, \cdots, 14$; and similarly for the other rows. This determinant does not vanish identically. For if we make the substitutions $\mathbf{a}_8 = \mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a}_9 = \mathbf{a}_2 + \mathbf{a}_3$, $\mathbf{a}_{10} = \mathbf{a}_1 + \mathbf{a}_3$, $\mathbf{a}_{11} = \mathbf{a}_1 + \mathbf{a}_7$, $\mathbf{a}_{12} = \mathbf{a}_2 + \mathbf{a}_7$, $\mathbf{a}_{13} = \mathbf{a}_3 + \mathbf{a}_7$, and $\mathbf{a}_{14} = \mathbf{y}$, an arbitrary vector, the determinant becomes [3]

$$\begin{vmatrix}
\substack{(x1452)\\(12367)}; & 0\ ; & P_1\ ; & P_2\ ; & P_3\ ; & P_4\ ; & P_5 \\
0\ ; & \substack{(x3562)\\(12473)}; & \substack{(x3561)\\(12473)}; & 0\ ; & 0\ ; & P_6\ ; & P_7 \\
0\ ; & \substack{(x3462)\\(12573)}; & \substack{(x3461)\\(12573)}; & 0\ ; & 0\ ; & P_8\ ; & P_9 \\
0\ ; & 0\ ; & 0\ ; & \substack{(x5671)\\(12347)}; & \substack{(x5672)\\(12347)}; & \substack{(x5673)\\(12347)}; & P_{10} \\
0\ ; & 0\ ; & 0\ ; & \substack{(x4671)\\(12357)}; & \substack{(x4672)\\(12357)}; & \substack{(x4673)\\(12357)}; & P_{11} \\
0\ ; & 0\ ; & 0\ ; & \substack{(x4571)\\(12367)}; & \substack{(x4572)\\(12367)}; & \substack{(x4573)\\(12367)}; & P_{12} \\
0\ ; & 0\ ; & 0\ ; & 0\ ; & 0\ ; & 0\ ; & P_0
\end{vmatrix}$$

where $P_1, \cdots P_{12}$ denote elements which are of no consequence, but $P_0 = (x456y)\ (1237y)$. It is evident that this determinant factors into a number of linear determinants together with the two-row determinant

$$\begin{vmatrix} (x3562); & (x3561) \\ (x3462); & (x3461) \end{vmatrix}$$

and the three-row determinant

$$\begin{vmatrix} (x5671); & (x5672); & (x5673) \\ (x4671); & (x4672); & (x4673) \\ (x4571); & (x4572); & (x4573) \end{vmatrix} \qquad (53)$$

[3] For compactness in printing this determinant, the two factors of each element are set one above the other. Thus the element in the first row and first column is

$$(x1452)\ (12367)$$

and similarly for the other factored elements.

The two-row determinant does not vanish identically since it does not vanish when $\mathbf{x} = \mathbf{a}_2 + \mathbf{a}_4$. Nor does the three-row determinant vanish identically, for if $\mathbf{x} = \mathbf{a}_4 + \mathbf{a}_1$ it becomes (45671) times the two-row determinant

$$\begin{vmatrix} (14672); & (14673) \\ (14572); & (14573) \end{vmatrix}$$

which, in turn, does not vanish if we let $\mathbf{a}_1 = \mathbf{a}_2 + \mathbf{a}_5$. Hence our original determinant does not vanish identically.

To complete the factorization, consider the minor of the leading element in (53), namely

$$(x4672)\,(x4573) - (x4572)\,(x4673); \tag{54}$$

and let $\mathbf{x}$ be expanded in terms of the five vectors $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$, and $\mathbf{a}_i$,—

$$\mathbf{x}(23457) - \mathbf{a}_2(x3457) + \mathbf{a}_3(x2457) - \mathbf{a}_4(x2357) + \mathbf{a}_5(x2347) - \mathbf{a}_7(x2345) = 0 \tag{55}$$

and multiplying by $[x467]$

$$-(x4672)\,(x3457) + (x4673)\,(x2457) + (x4675)\,(x2347) = 0 \tag{56}$$

whence it is evident that (54) contains the factor $(x4567)$. In a similar manner it may be shown that the other minors of the elements of the first row in (53) contain the same factor. Thus the determinant contains this factor. In the same way we may show that the minors of any other row contain a common factor, hence the determinant is the product of three linear factors. Again, the same process shows that (52) is a product of two linear factors. The factorization is therefore complete, and the adventitious factor of our original determinant is here, as in the former case, a mere product of determinants linear in the points which enter into them.

7. Rule for Constructing these Determinants.

It remains to indicate how, in general, the $\mathbf{b}$'s may be chosen so that the determinant contemplated in theorem V shall not vanish identically. We have seen that the order q of this determinant is $n[(N-1), K] - N + 2$. Let q' be the number analogous to q but associated with polynomials of degree less by one, that is

$$q' = n[N-1, K-1] - N + 2 \tag{57}$$

The elements of the determinant are of the form $\mathbf{b}\cdot\mathbf{a}_j E(\mathbf{a}_j)$. A necessary condition that the determinant shall not vanish identically is

that not more than $q' + N - 2$ of the E's shall be polynomials of the same standard set. For when several E's belong to the same standard set they are functions of a set of $m + 1$ of the **a**'s not including $\mathbf{a}_j$. Since any element of the determinant contains every one of a set of $m + N - 2$ of the **a**'s (exclusive of $\mathbf{a}_j$), and also contains **x**, it follows that the corresponding **b**'s will possess $N - 3$ of the **a**'s in common. That is, they have in common $N - 2$ arbitrary vectors not belonging to the reference system of the E's which occur in the same elements. Now by theorem IV, if $q' + 1$ of the E's belong to a standard set they satisfy an identity $\Sigma L(\mathbf{a}_j)E(\mathbf{a}_j) = 0$, but the L's are functions of those **a**'s only which occur in the E's. By writing for the E's an identity of the form (32) it is evident that even when the L's contain $N - 2$ arbitrary vectors we shall have $q' + 1 + (N - 2)$ of the E's (which now correspond to the C's of (32)), connected by a relation $\Sigma LE = 0$. Hence only $q' + N - 2$ of the E's can belong to the same standard set, as was to be shown.

We are now in a position to prove that the **b**'s may in general be selected by the following rule:

Rule for choice of b's. Select any $m + N - 2$ from the given set of $n - 1$ vectors $\mathbf{a}_1 \cdots \mathbf{a}_{n-1}$. Choose $q' + N - 2$ polynomials $E_1, E_2, \cdots$ out of the standard set of degree $K - 1$ based on $m + 1$ of these vectors, which we may number from 1 to $m + 1$.

Make a second selection of $m + 1$ **a**'s by omitting $\mathbf{a}_1$ from the first selection and adjoining $\mathbf{a}_{m+2}$, choose $q' + N - 3$ polynomials of the new standard set based on this second selection of **a**'s.

If we have not yet q polynomials, make a third selection of $m + 1$ **a**'s by omitting $\mathbf{a}_{m+2}$ from the second selection and adjoining $\mathbf{a}_{m+3}$; choose $q' + N - 3$ polynomials from a third standard set based on this third selection of **a**'s.

If we have not yet q polynomials, proceed in a similar manner to form new groups until q polynomials have been obtained: every selection of **a**'s consists of $\mathbf{a}_2, \mathbf{a}_3, \cdots \mathbf{a}_{m+1}$ together with one other. Each of the **b**'s contains every **a** not in its own 'E' but included in the original choice of $m + N - 2$ of the **a**'s.

We have to prove that the determinant whose elements are $\mathbf{b} \cdot \mathbf{a}_j$ $E(\mathbf{a}_j)$ does not vanish identically, and that the rule is always possible.

We note that, by the rule, $\mathbf{a}_1$ occurs only in the E's numbered from 2 to $q' + N - 2$ inclusive. Hence $\mathbf{a}_1$ occurs in all the **b**'s except those so numbered. If the determinant vanishes identically a set of numbers $c_1 \cdots c_q$ exists such that $\Sigma c\mathbf{b} \cdot \mathbf{a}_j E(\mathbf{a}_j) = 0$, the c's being independent of $\mathbf{a}_j$ and not all zero. Now let $\mathbf{a}_j = \mathbf{x} + g\mathbf{a}_1$ where g is

an arbitrary scalar. All terms vanish except those from 2 to $q'+N-2$. The equation takes the form $\Sigma c_i \mathbf{b}_i \cdot \mathbf{a}_1 E_1(\mathbf{x} + g\mathbf{a}_1) = 0$ where i runs from 2 to $q' + N - 2$.

Consider the quantity $E_i\,(\mathbf{x} + g\mathbf{a}_1)$. It is a polynomial in g whose absolute term is $E_i(\mathbf{x})$. Since the above equation is to hold for all values of $\mathbf{a}_j$ it must hold for all values of g. Hence the sum of the absolute terms must vanish, that is $\Sigma c_i \mathbf{b}_i \cdot \mathbf{a}_i E_i(\mathbf{x}) = 0$ where i runs through the $q' + N - 3$ values above noted. But this is impossible unless the c's which have these subscripts are all zero: for the equation now has the form $\Sigma L(\mathbf{x})E(\mathbf{x}) = 0$, the only $\mathbf{a}$ common to L and E is $\mathbf{a}_1$, whence not fewer than $q' + N - 2$ E's satisfy such a relation, by the corollary to theorem IV.

Similarly we note that $\mathbf{a}_{m+2}$ occurs only in the second group of E's, and prove in the same way that all c's corresponding to this group must be zero, and so on, till finally only c_1 remains. Hence c_1 is also zero. This is contrary to hypothesis that the c's be not zero. Hence the determinant does not vanish identically.

To show, finally, that the rule is always possible, we note, first, that the rule observes the plan of grouping the E's so that no more than $q' + N - 2$ belong to a standard set, E_1 being common to all the groups. The first group requires $m + 1$ of the $\mathbf{a}$'s. Each new group requires one additional $\mathbf{a}$. We have $m + N - 2$ $\mathbf{a}$'s available for selection. It therefore suffices to show that the number of groups is not greater than $N - 2$. We have, in other words to prove the inequality

$$q \gtreqless (N-2)\,(q' + N - 3) + 1 \tag{58}$$

Putting for q and q' their values in terms of N and K this becomes

$$\frac{(K+1)\,(K+2)\cdots(K+N-2)}{(N-2)!} - N + 2 \gtreqless$$

$$\frac{(N-2)K(K+1)\,(K+2)\cdots(K+N-3)}{(N-2)!} - (N-2)$$

which simplifies to $0 \gtreqless (N-2)K(K+N-3)$. This holds when N is 3 or more, and binary forms were above explicitly excluded from consideration. The method is thus completely established.

In agreement with these results we note that when $N = 3$ we have but one group, as exemplified by (38). When $N = 4$ we shall always have two groups, but they are not both complete. If the first group be always completed, a smaller and smaller proportion of the E's of

the second group is needed as K increases. When $N = 5$ both groups are complete only in the case $K = 2$. When $N = 6$ we shall need part of a third group only when $K = 2$ or 3. These and many other facts can be seen at once by a table of the numbers used in the discussion:

Case $N = 3$:

$K =$	1	2	3	4	5	6	7	
$n =$	3	6	10	15	21	28	36	
$m =$	0	2	5	9	14	20	27	$= n[N,K-1]-1$
$m + N - 2 =$	1	3	6	10	15	21	28	
$q =$	1	2	3	4	5	6	7	$= n - m - N + 1$
$q' =$		1	2	3	4	5	6	

and $q' + N - 2 = q$ so that one group suffices in 3-space.

Case $N = 4$:

$K =$	1	2	3	4	5	6	7
$n =$	4	10	20	35	56	84	120
$m =$	0	3	9	19	34	55	83
$m + N - 2 =$	2	5	11	21	36	57	85
$q =$	1	4	8	13	19	26	34
$q' =$		1	4	8	13	19	26
$q' + N - 2 =$		3	6	10	15	21	28

so that $q' + N - 2 > q/2$ for any K.

Case $N = 5$:

$K =$	1	2	3	4	5	6	7
$n =$	5	15	35	70	126	210	330
$m =$	0	4	14	34	69	125	209
$m + N - 2 =$	3	7	17	37	72	128	212
$q =$	1	7	17	32	53	81	117
$q' =$		1	7	17	32	53	81
$q' + N - 2 =$		4	10	20	35	56	84

whence again $q' + N - 2 > q/2$ for any K.

Case $N = 6$:

$K =$	1	2	3	4	5
$n =$	6	21	56	126	252
$m =$	0	5	20	55	125
$m + N - 2 =$	4	9	24	59	129
$q =$	1	11	31	66	122
$q' =$		1	11	31	66
$q' + N - 2 =$		5	15	35	70

so that $q' + N - 2 > q/2$ except when $K = 2$ or 3.

It is interesting to note that $q - q' = n[N - 2, K]$, serving as a check on the correctness of the table.

8. Special Cases of the General Identity: Method of Reduplication.

A third method of obtaining identities from the fundamental identities (5) or (9) depends on the use of a function of two variable points. I therefore call it the method of reduplication. An example of the use of this method for the case $N = 3$, $K = 2$ I have elsewhere given.[4] Let $F(\mathbf{x}, \mathbf{y})$ be a function (vector or scalar) of two points $\mathbf{x}$ and $\mathbf{y}$. Applying (5) or (9) we have

$$F(\mathbf{x}, \mathbf{y})C_0 + F(\mathbf{a}_1, \mathbf{y})C_1 + F(\mathbf{a}_2, \mathbf{y})C_2 + \cdots + F(\mathbf{a}_n, \mathbf{y})C_n = 0 \quad (59_0)$$

and by giving to $\mathbf{y}$ in succession the values $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$ we have the n further equations

$$F(\mathbf{x}, \mathbf{a}_1)C_0 + F(\mathbf{a}_1, \mathbf{a}_1)C_1 + F(\mathbf{a}_2, \mathbf{a}_1)C_2 + \cdots + F(\mathbf{a}_n, \mathbf{a}_1)C_n = 0 \quad (59_1)$$
$$F(\mathbf{x}, \mathbf{a}_2)C_0 + F(\mathbf{a}_1, \mathbf{a}_2)C_1 + F(\mathbf{a}_2, \mathbf{a}_2)C_2 + \cdots + F(\mathbf{a}_n, \mathbf{a}_2)C_n = 0 \quad (59_2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F(\mathbf{x}, \mathbf{a}_n)C_0 + F(\mathbf{a}_1, \mathbf{a}_n)C_1 + F(\mathbf{a}_2, \mathbf{a}_n)C_2 + \cdots + F(\mathbf{a}_n, \mathbf{a}_n)C_n = 0 \quad (59_n)$$

We now let $\mathbf{x} = \mathbf{y}$ in (59_0), then multiply these $n + 1$ equations by $C_0, C_1, C_2, C_3, \cdots C_n$, respectively, and add. For simplicity of notation we write $\mathbf{a}_0 = \mathbf{x}$. The result is

$$\Sigma F(\mathbf{a}_i, \mathbf{a}_j)C_1C_j = 0 \quad (60)$$

where both i and j run from 0 to n inclusive. From this identity, by giving to the function F various forms, scalar or vector, a great number of relations connecting the C's may be obtained. The method may also be applied, by triplication, to functions of three points, and so on.

[4] The Axes of a Quadratic Vector, Proc. Amer. Acad. Arts and Sci., vol. 56, No. 9. June 1921, p. 333.